2017-2018 Puzzle Contests

Solutions for Contest #5

Parents and Grandparents Puzzle Solutions:

1. **In some city, anyone 40 years or older always tells** the truth and anyone under 40 never tells the truth. A census taker knocks on the married couple's door. The husband opens the door. 'I am the census taker,' says the visitor,' and I need information about you and your wife. Which, if either of you, are under the age of 40?' 'We are both under 40!' said the husband angrily as he slammed the door. How did the census taker sort out the facts? Why? **(25 pts)**

Answer: The census taker sorted out the fact as the husband is younger than 40, his wife is older than 40

Proof: Let h and w denote the ages of the husband and wife respectively. Then $h < 40$, otherwise the husband would have been telling the truth, which he cannot do unless his is at least 40. But if he is lying, he must not be lying about his own age. He must therefore be lying about his wife's age. We are forced to conclude that $h < 40$ and $w > 40$.

2. **You and your spouse invite four other couples to a spring party**. During the course of the conversation, it is discovered that, prior to the party, each person except you was acquainted with a different number of the people present. Assuming the acquaintance relationship is symmetric (i.e., if you are acquainted with someone, that person is also acquainted with you), then how many people did your spouse know prior to the party? How many people did you know? **(30 pts)**

Answer : I knew 5 people, my wife knew five also

Solution: Start with the assumption that everybody knows their own spouses -- which means that everybody there knew at least one person. Discounting yourself, everyone knows a different number of people, which means that (again, discounting yourself) one person knows one, one person knows two, one person knows three, etc., up to one person who knows nine people (everybody else). Number the people (besides yourself) according to how many people they know, so that person 1 is the one who knows one person, person 2 is the one who knows two people, etc. Now pair up people with their spouses. If person 9 knows everybody else, s/he must be the only person who knows person 1, because person 1 only knows one person. So they must be married.

Person 8 knows everybody except for person 1. Person 2 therefore knows person 8 and person 9. Person 9 is married to person 1, so person 2's spouse must be person 8. Person 7 knows everybody except for persons 1 and 2. Person 3 therefore knows persons 7, 8, and 9. Persons 8 and 9 are married to persons 2 and 1 respectively, so person 3's spouse must be person 7. Person 6 knows everybody except for persons 1, 2, and 3. Person 4 therefore knows persons 6, 7, 8, and 9. The only one of those not yet paired up is person 6, so person 4 and person 6 must be married. This leaves person 5, who knows everyone except persons 1, 2, 3, and 4. These five people, therefore, must be persons 6, 7, 8, 9, and you. Since you are the only one of these five not yet paired up, person 5 must be your spouse. So your spouse knew five people prior to the party. The above also determines that the people who know you are persons 5, 6, 7, 8, and 9. So you knew five people prior to the party also.

3. The cubic polynomials $p(x)$ and $q(x)$ satisfy the following conditions:

- $p(1) = q(2)$
- $p(3) = q(4)$
- $p(5) = q(6)$
- $p(7) = q(8) + 13.$

Find $p(9) - q(10)$. **(45 pts)**

Answer: 52

Solution: Note that $q(x+1)$ is a cubic polynomial as is $q(x)$. Consider the polynomial $r(x) = p(x) - q(x+1)$ and note that

1. $r(x)$ is a cubic polynomial or lees than three as the difference of two cubic polynomials
2. $r(1) = p(1) - q(2) = 0, r(3) = p(3) - q(4) = 0, p(5) - q(6) = 0$ according to the condition of the problem i.e. the numbers $1, 3$ and 5 are the roots of $r(x)$.
3. $r(x) = A(x-1)(x-3)(x-5), where\ A\ is\ a\ constant$ according to the Remainder Theorem and the Main Theorem of Algebra
4. $r(7) = A(7-1)(7-3)(7-5) = p(7) - q(8) = 13$ according to the condition of the problem. Hence $A = \frac{13}{48} \neq 0$ i.e $\deg\big(r(x)\big) = 3$
5. $p(9) - q(10) = r(9) = \frac{13}{48}\ (9-1)(9-3)(9-5) = \frac{13}{48}\ 8 \cdot 6 \cdot 4 = 52$